LOWER BOUNDING THE CAPACITATED
LOT SIZE PROBLEM

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Abstract

The capacitated lot size problem (CLSP) concerns the attempt to schedule in detail while taking aggregate resource constraints into consideration. Aggregate resource management under conditions of seasonal demand has been researched extensively, as has the uncapacitated lot size problem under various assumptions of stationarity, costs, and problem size. However, the existence of a fixed charge (the set-up cost) has precluded the determination of optimal solutions for combined problems.

The objective of this paper is to consider the CLSP for fixed resources, and to expand upon the lower bounding technique first suggested by Fisher (19) for the optimal solution to the CLSP. It will be seen that the bounding function is continuous and concave, a promising structure for analytical determination of the strongest bound and for good solutions to the CLSP.

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Introduction

Over the past few decades, the use of mathematical models to solve complex resource allocation problems has become common. Progress over the years has focused both upon the application of new analytical techniques and upon exploiting the structure of a particular problem to obtain superior numerical results. It is the latter which concerns us here.

The capacitated lot size problem (CLSP) concerns the attempt to schedule in detail while taking aggregate resource constraints into consideration. Aggregate resource management under changing demand has been researched thoroughly (1) (12) (14) (16) (26) (27) as has the uncapacitated lot size problem under various assumptions of stationarity, costs and problem size (2) (5) (13) (18) (22) (24) (28). However, the existence of a fixed charge has precluded effective solution of the combined problem. Strictly heuristic procedures (21) and strong approximations (8) now exist. However, the degree of their success is dependent upon the problem configuration.

The objective of this paper is to expand upon the lower bounding technique for the CLSP first suggested by Fisher (19). It will be seen that the bounding function is continuous and concave, a promising structure for analytical determination of the strongest bound and for good solutions to the CLSP.

The plan of this paper is as follows. First, a formulation of the CLSP is defined. Then, a related problem structure is reviewed and discussed -- the capacitated network. The network approach is used to derive the form of a lower bounding function. Discussion of some empirical results and some extensions to the lower bounding procedure complete the presentation. A companion method for establishing upper bounds is suggested in Appendix B, using methods described in Appendix C.

Statement of the Problem

Consider a production facility with limited fixed resources, with a requirement to produce I different products over a horizon of T periods. The demand for each product is known with certainty and demand must be satisfied in the period it occurs. Backorders are not permitted. There is a fixed setup charge incurred when a resource is changed over to different product. The setup charges are independent of subsequent levels of production and of the prior production configuration of the facility. Also, a loss in productive time ("down-time") is incurred during product changeover.

Variable production and holding costs are linear and stationary within products, though they will vary among products. The cost function is thus concave, consisting of the fixed charge and the linear production and holding costs.

The objective is to minimize the total setup, production, and holding costs subject to the demand and capacity constraints.

Mathematically, the problem is stated:

1.0 $$\text{Min} \sum_i \sum_t [S_i \delta(P_{it}) + V_i P_{it} + h_i I_{it}]$$

Subject to:

1.1 $$I_{i,t-1} + P_{it} - I_{it} = d_{it} \qquad i = 1,\ldots,I \quad t = 1,\ldots,T$$

1.2 $$\sum_i [r^{\delta}_{ik} \delta(P_{it}) + r^{P}_{ik} P_{it}] \leq R_{kt} \qquad k = 1,\ldots,K \quad t = 1,\ldots,T$$

1.3 $$P_{it}, I_{it} \geq 0$$

1.4 $$\delta(P_{it}) = \begin{cases} 0 & \text{if } P_{it} = 0 \\ 1 & \text{if } P_{it} > 0 \end{cases}$$

where:

P_{it} = production of product i in period t.

I_{it} = inventory level of product i at the end of period t.

$\delta(P_{it})$ = variable assigning setup cost for product i to period t when $P_{it} > 0$.

V_i = per unit production cost for product i.

h_i = per period unit holding cost for product i.

S_i = setup cost for product i.

d_{it} = demand for product i in period t.

r^{δ}_{ik} = capacity absorption for one setup of product i on resource k.

r^{P}_{ik} = per unit capacity absorption of product i on resource k.

R_{kt} = the level of resource k available in period t.

K = the number of resources.

T = the number of periods (horizon).

I = the number of products.

This problem will be referred to as Problem One.

There are two characteristics of the problem which will be expanded upon. First, if no down-time occurs during product changeover, i.e., $r^{\delta}_{ik} = 0$, the constraints form a convex polyhedron. Since the objective function is concave, the optimal solution will occur at one or more of the extreme points of the polyhedron. This problem structure is called "the fixed charge problem" (FCP) and a review of this subject is contained in Appendix C.

Second, and of direct interest here, is that constraints (1.1) form a single source, single sink transhipment network, a structure that has been exploited by Wagner and Whitin (WW)(28) and Zangwill (32) for solving the unconstrained lot size problem. This structure will be exploited later in calculating lower bounds for the CLSP.

The CLSP as a Network Problem

Those familiar with the dynamic lot size problem will have recognized that constraints (1.1) plus objective function 1.0 form a single-source, single-sink transhipment network which may be solved by dynamic programming in the efficient manner suggested by Wagner and Whitin (WW) and extended upon by Zangwill (32).

The Fundamental Postulate of WW reduces the search space of extreme points considerably: "There exists an optimal program such that in any period t the facility need not produce and enter the period with previous periods' production." (28)

Proof of the Fundamental Postulate: (28) Suppose an optimal program suggests both to produce in period t and to bring inventory into the period. Then it is no more costly to reschedule the production of that inventory by including it in the production for that period, for the alteration does not incur any additional setup cost and does save the holding cost $h_i I_{it}$.

The network interpretation of the Postulate would read: "Specifically, an optimal flow* belongs to a class of flows known as extreme flows. Extreme flows relate to the extreme points of the polyhedral region described by constraints (1.1). That is, a node can receive material from at most one other node." (32)

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* A shipping pattern which minimizes total shipping cost.

Referring to Figure 1, it can be seen that "material" can come from the previous transhipment node (inventory) or from the single source (production). The last period forms the single sink.

In applying constraints (1.2), the lot size model is altered considerably, forming a capacity constrained network.

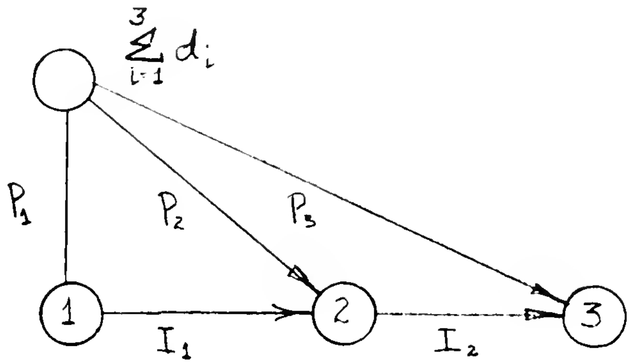

FIGURE 1

An Approximate Solution to the CLSP

We should mention before continuing the most successful large-scale approximation of the problem to date. Expanding the work of Manne (19) and Dzielinski, Baker, and Manne (7), Dzielinski and Gomory (8) have solved very large versions of Problem One for general resource constraints in various configurations of up to 10 periods, 900 products, and 2 resources. The formulation is an approximation, but a very good one if the product of the number of resources and time periods (KT) is much less than the number of items (I). However, if KT is approximately equal to I, the formulation gives indeterminate answers. It is this weakness of the Dzielinski and Gomory model which has motivated this paper.

Dzielinski and Gomory's major contribution was the adoption of decomposition linear programming and sequential column generation to reduce the effective problem size. Of particular interest here is their method of selecting columns for entry into the basis. Using artificial costing based on the duals for the current basis, columns were generated as needed so that only promising columns were included in the formulation. The WW algorithm for non-stationary costs was the vehicle for column generation. It will be seen that the lower bounding technique employs a markedly similar approach.

We now turn to the subject of optimality and lower bounding.

Optimal Solution of Resource Constrained Network Scheduling Problems

Fisher (9) in his thesis (of the above title) applied his techniques to the job shop scheduling environment. The objective was to determine a set of start times which minimized some function of task completion times. In solving the general resource constrained job shop problem, he determined a family of lower bounds on the optimal objective value.

Of interest to us here is his extension (not proved in his thesis) of the lower bounding technique to the capacitated lot size problem.

Lower Bounds for the CLSP

Fisher determines his lower bounds with a subproblem formed by placing the resource constraints in the objective function. By applying artificial costs (Lagrange multipliers) to the newly formed subproblem and solving the resulting problem for unlimited resources, a family of lower bounds is produced.

Define Problem A:

3.0 Minimize: $f(x) = \underline{C}\ \underline{x} + \underline{D}\ \underline{\delta}(x)$

3.1 Subject to: $\underline{A}_1\underline{x} = \underline{b}_1$

3.2 $\underline{A}_2\underline{x} + \underline{B}\delta(\underline{x}) \leq b_2$

3.3 $x \geq 0$

3.4 $\delta(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$

Define Problem B:

4.0 Minimize: $g(x,\pi) = \underline{C}\ \underline{x} + \underline{D}\delta(\underline{x}) + \underline{\pi}\ (\underline{A}_2\ \underline{x} + \underline{B}\delta(\underline{x}) - \underline{b}_2)$

4.1 Subject to: $\underline{A}_1\underline{x} = \underline{b}_1$

4.2 $x \geq 0$

4.3 $\delta(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$

Theorem 1: For all $\underline{\pi} \geq 0$, any optimal solution to Problem B represents a lower bound on all feasible solutions to Problem A.

Proof: Define x' as any feasible solution to Problem A.

Define $x^* = x^*(\pi)$ as the optimal solution to Problem B for any $\underline{\pi} \geq 0$.

Fact: If x' is feasible in A, it is also feasible in B.

Therefore,

$$g(x^*,\pi) \leq g(x',\pi) = Cx' + D\delta(x') + \underline{\pi}\ (A_2x' - B\delta(x') - b_2)$$

But $\underline{\pi} \geq 0$ and x' feasible in Problem A imply

$$g(x',\pi) \leq Cx' + D\delta(x') = f(x')$$

or, in summary,

$$g(x^*,\pi) \leq f(x')$$

QED

Lower Bounding the CLSP

Adapting Fisher's general formulation to our Problem One, we represent the uncapacitated lot size problem as $A_1x = b_1$, and the fixed capacity constraints as $A_2x + B\delta(x) = b_2$.

Problem A:

1.0 $$\text{Minimize } Z_A = \sum_i \sum_t [S_i\delta(P_{it}) + V_iP_{it} + h_iI_{it}]$$

Subject to:

1.1 $$I_{i,t-1} + P_{it} - I_{it} = d_{it} \qquad \begin{array}{l} i = 1,\ldots,I \\ t = 1,\ldots,T \end{array}$$

1.2 $$\sum_i [r^{\delta}_{ik}\delta(P_{it}) + r^{P}_{ik}P_{it}] \leq R_{kt} \qquad \begin{array}{l} k = 1,\ldots,K \\ t = 1,\ldots,T \end{array}$$

1.3 $$P_{it}, I_{it} \geq 0$$

1.4 $$\delta(P_{it}) = \begin{cases} 0 \text{ if } P_{it} = 0 \\ 1 \text{ if } P_{it} > 0 \end{cases}$$

Problem B:

5.0 $$\text{Minimize: } Z_B(\pi) = \sum_i \sum_t [(S_i + \sum_k \pi_{kt}r^{\delta}_{ik})\delta(P_{it}) + (V_i + \sum_k \pi_{kt}r^{P}_{ik})P_{it} + h_iI_{it}] - \underline{\pi R}_{KT}$$

5.1 Subject to:

$$I_{i,t-1} + P_{it} - I_{it} = d_{it} \qquad \begin{array}{l} i = 1,\ldots,I \\ t = 1,\ldots,T \end{array}$$

5.2 $$P_{it},\ I_{it} \geq 0$$

5.3 $$\delta(P_{it}) = \begin{cases} 0 \text{ if } P_{it} = 0 \\ 1 \text{ if } P_{it} > 0 \end{cases}$$

The perceptive reader will have noted that the existence of non-zero components for $\underline{\pi}$ will impose a nonstationary cost structure upon the WW model. However, as stated by Wagner, the algorithm still holds for non-stationary costs (34).

Fundamental Postulate (adjusted): "There exists an optimal program such that in any period t the facility need not produce and enter the period with previous periods' production if the marginal cost curves are not necessarily identical in all periods. Non-negative (non-identical) setup costs may be included."

This result allows the use of the efficient WW algorithm for the solution to Problem B.

Since Problem B is a multi-product WW lot size model, the optimal solution $Z_B^*(\pi)$ to B for given $\underline{\pi}$ will be the sum of the individual WW solutions less $\underline{\pi}\ \underline{R}_{KT}$.

$$Z_B^*(\underline{\pi}) = \sum_{i=1}^{I} Z_{B_i}^* (\underline{\pi}) - \underline{\pi}\ \underline{R}_{KT}$$

where:

$$Z_{B_i}^* (\underline{\pi}) = \min_{P_{it}, I_{it}} \sum_{t=1}^{T} [(S_i + \sum_k \pi_{kt} r_{ik}^{\delta}) \delta(P_{it}) + (V_i + \sum_k \pi_{kt} r_{ik}^{P}) P_{it} + h_i I_{it}]$$

$i = 1,\ldots,I$

Subject to: $I_{i,t-1} + P_{it} - I_{it} = d_{it}$ $\quad t = 1,\ldots,T$

$$P_{it}, I_{it} \geq 0$$

$$\delta(P_{it}) = \begin{cases} 0 \text{ if } P_{it} = 0 \\ 1 \text{ if } P_{it} > 0 \end{cases}$$

The challenge remains to determine the "best" π-vector that produces the strongest lower bound, i.e., to find

$$Z_B^*(\underline{\pi}^*) = \max_{\underline{\pi}} Z_B^*(\underline{\pi})$$

Limited empirical experience suggests that strong lower bounds can be achieved by applying arbitrary values for $\underline{\pi}$. But a more powerful result is now derived.

The Concavity and Continuity of $Z_B^*(\pi)$

Since we wish to determine max $Z_B^*(\underline{\pi})$, the concavity and continuity of $Z_B^*(\pi)$ is of interest.

Theorem 2: Define $f(x,\pi)$ as a concave function of π for all x and let the function $Z_B^*(\pi)$ be defined by

$$Z_B^*(\pi) = \min_{\substack{Ax=b \\ x\geq 0}} f(x,\pi)$$

$$= \min_{\substack{Ax=b \\ x\geq 0}} \sum_{it} [(S_i + \sum_k \pi_{kt} r_{ik}^{\delta})\delta(P_{it}) + (V_i + \sum_k \pi_{kt} r_{ik}^{P})P_{it} + h_i I_{it}] - \underline{\pi}\, \underline{R}_{KT} \quad .$$

Then $Z_B^*(\pi)$ is concave and continuous for all $\underline{\pi}$.

Proof: Let $\pi = \alpha\pi_1 + (1-\alpha)\pi_2, \quad 0 \leq \alpha \leq 1$

Since $f(x,\pi) \geq \alpha f(x,\pi_1) + (1-\alpha) f(x,\pi_2)$

for all x, clearly

$$Z_B^*(\pi) = \min_{\substack{Ax=b \\ x\geq 0}} f(x,\pi) \geq \min_{\substack{Ax=b \\ x\geq 0}} [\alpha f(x,\pi_1) + (1-\alpha) f(x,\pi_2)]$$

$$\geq \min_{\substack{Ax=b \\ x\geq 0}} f(x,\pi_1) + \min_{\substack{Ax=b \\ x\geq 0}} (1-\alpha) f(x,\pi_2)$$

$$= \alpha Z_B^*(\pi_1) + (1-\alpha) Z_B^*(\pi_2)$$

where the last inequality follows from obvious properties of the minimization operator.

Clearly, $f(x,\pi)$ as defined in the hypothesis satisfies the concavity statement. Therefore, $Z_B^*(\pi)$ is concave for all π and thus it is also continuous for all π.

QED

Implications of the Concavity and Continuity of $Z_B^*(\pi)$

Since $Z_B^*(\pi)$ is concave and continuous in all π, it can be inferred that algorithmic procedures for determining $\underline{\pi}^*$ such that $Z_B^*(\underline{\pi}^*) = \max_{\pi} Z_B^*(\underline{\pi})$ are possible. In view of the greatly reduced search space presented by the multi-item WW model and the relatively few components of π, i.e. KT, search procedures for (at least) modest problems should be practicable. Further, the piecewise linearity of $Z_B^*(\pi)$ encourages the use of hill-climbing or gradient methods. (See Wilde and Beightler [31]).

Empirical Results on the Lower Bound (Problem B)

Limited experience exists on the behavior of Problem B. An interactive code in PL1 was the vehicle for experimentation. The piecewise linear concavity of $Z_B^*(\pi)$ was observed and the application of arbitrary π-vectors to determine a strong lower bound was successful.

Appendix A contains the results for a four-product, five-period, single resource problem. To clarify the results, here is a summary of the bounding procedure.

1. The unconstrained WW I-item solution was obtained with $\pi = 0$ providing an absolute lower bound to Problem A (Problem One) of $1930.00.

2. The result is infeasible in Problem A in periods 1 and 4. Problem B was then resolved using a π-vector of $\underline{\pi} =$ (3, 0, 0, .6, 0) which applied penalty costs of $3.00 and $ .60 respectively to capacity used in periods 1 and 4. The value of $Z_B^*(\pi)$ increases to $2024.60. This sum represents a lower bound for the optimum solution to Problem A.

Note that the purpose of Problem B is to provide a lower bound on the value of Problem A. However, it can happen that the production plan suggested in Problem B for $\underline{\pi} \geq \underline{0}$ is also feasible in Problem A. Such was the case with $\underline{\pi}$ = (3, 0, 0, .6, 0). The feasible allocation evaluation in Problem A became $2090.00 providing an <u>upper</u> bound on the optimum solution.*

* The best solution value found was $2045.00. Since this is a pure strategy there might exist a π-vector which would produce it in Problem B. The vector was not found.

Extensions to the Lower Bounding Technique

The lower bounding technique may be extended to evaluate arbitrary configurations of the production system. Suppose it were specified that production must occur in certain periods not considered in the optimal unconstrained WW solution to Problem B. The absolute lower bound would increase, and the maximum lower bound $Z_B^*(\pi^*)$ would reflect a value with respect to the configuration specified.

An example is provided in Appendix A, pp. 29 and 30 . Note that non-zero production has been specified for product 1 for every period. The absolute lower bound (at $\pi = 0$) has increased to \$2215.00 from \$1930.00 and, after application of $\pi = (3, 0, 0, .6, 0)$, the lower bound becomes \$2289.00.

The purpose of this discussion becomes clearer if we assume that the configuration in the example was defined during an implicit search algorithm. Since we already have an upper bound of \$2090.00, there would be no need to continue exploring this branch of the tree.*

We cannot assure the generation of an upper bound through Problem B. For efficient branching, good feasible solutions (upper bounds) must be found. A simple but general method is suggested in Appendix B, using the fixed charge heuristics discussed in Appendix C.

* The best solution of \$2045.00 with a lower bound of \$2043.20 was generated in this manner.

Summary and Conclusions

A method for determining strong lower bounds for the optimal objective function of the capacitated lot size problem has been studied. The bounding function was found to be concave and continuous, suggesting algorithmic procedures for determining the strongest bound are possible.

The bounding procedure may be useful for implicit search algorithms since lower bounds may be established for each node. A companion method of determining upper bounds enhances the future of implicit search for the CLSP. (See Appendix B.)

Finally, if approximate solution procedures to the CLSP are employed, the available lower bound will provide a measure of optimality. In the past, this measure has not been explicitly available.

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APPENDIX A

Sample Problem

Consider the four-product, five-period problem configuration with one fixed resource. The cost structure among products is identical, as are the demand profiles. The differentiation among products occurs only in the demand rate, which is 1: 2: 3: 4.

Setup Cost - S_i = \$130 per setup $i = 1, \ldots, 4$

Variable cost - V_i = 0 per unit   "

Setup Capacity Absorption - r_i^{δ} = 1 per setup   "

Variable cap. Absorption - r_i^{P} = 1 per unit   "

Holding Cost - h_i = 3 per unit per period   "

Capacity constraint - R_t = 140 $t = 1, \ldots, 5$

DEMAND:

	PERIOD				
PRODUCT	1	2	3	4	5
1	10	5	10	15	10
2	20	10	20	30	20
3	30	15	30	45	30
4	40	20	40	60	40

SOLUTION TO PROBLEM B WITH $\pi = 0$

	PRODUCT	PERIOD 1	2	3	4	5
1. Setups	1	1	0	0	1	0
	2	1	0	0	1	0
	3	1	0	1	1	0
	3	1	0	1	1	0
2. Unit Production	1	25	0	0	25	0
	2	50	0	0	50	0
	3	45	0	30	75	0
	4	60	0	40	100	0
3. Capacity Absorption	1	26	0	0	26	0
	2	51	0	0	51	0
	3	46	0	31	76	0
	4	61	0	41	101	0
	Total	184	0	72	254	0
	Infeasibilities	44	0	0	104	0

Lower Bound Value $1930.00

Cost per <u>Problem A</u> $1930.00

SOLUTION TO PROBLEM B WITH $\pi = (3,0,0,.6,0)$

	PRODUCT	PERIOD 1	2	3	4	5
1. Setups	1	1	0	1	0	0
	2	1	0	1	0	0
	3	1	1	0	1	0
	4	1	1	0	1	1
2. Unit Production	1	15	0	35	0	0
	2	30	0	70	0	0
	3	30	45	0	75	0
	4	40	60	0	60	40
3. Capacity	1	16	0	36	0	0
	2	31	0	71	0	0
	3	31	46	0	76	0
	4	41	61	0	61	41
	Total	119	107	107	137	41
	Infeasibilities	0	0	0	0	0

Lower Bound VAlue $2025.20

Cost per Problem A $2090.00 (an upper bound).

The π-vector was determined by increasing the value of π_t in periods 1 and 4 (the infeasible periods) until the "good" solution occurred.

THE BEST SOLUTION TO PROBLEM A (found by Specifying $(P_{42}) = 1$

And Applying $\underline{\pi} = (0,0,0,.6,0)$).

	PRODUCT	Period 1	2	3	4	5
1. Setups	1	1	0	1	0	0
	2	1	0	1	0	0
	3	1	0	1	1	0
	4	1	1	0	1	1
2. Unit Production	1	15	0	35	0	0
	2	30	0	70	0	0
	3	45	0	30	75	0
	4	40	60	0	60	40
3. Capacity Absorption	1	16	0	36	0	0
	2	31	0	71	0	0
	3	46	0	31	76	0
	4	41	61	0	61	41
	Total	134	61	137	137	41
	Infeasibilities	0	0	0	0	0

Lower Bound Value $2043.20

Cost Per <u>Problem A</u> $2045.00

THE AFFECTS OF FIXED CONFIGURATION ON PROBLEM B

(a) $P_{1t} > 0$ for $t = 1,\ldots T$

(b) $\underline{\pi} = 0$

	Product	PERIOD 1	2	3	4	5
1. Setups	1	1	1	1	1	1
	2	1	0	0	1	0
	3	1	0	1	1	0
	4	1	0	1	1	0
2. Unit Production	1	10	5	10	15	10
	2	50	0	0	50	0
	3	45	0	30	75	0
	4	60	0	40	100	0
3. Capacity Usage	1	11	6	11	16	11
	2	51	0	0	51	0
	3	46	0	31	76	0
	4	61	0	41	101	0
	Total	168	6	83	244	11
	Infeasibilities	28	0	0	104	0

Lower Bound Value $2215.00

Cost per Problem A $2215.00

THE EFFECTS OF FIXED CONFIGURATION ON PROBLEM B (Con'td)

(a) $P_{1t} > 0$, $t = 1, \ldots, T$

(b) $\underline{\pi} = (3, 0, 0, .6, 0)$

	PRODUCT	PERIOD 1	2	3	4	5
1. Setups	1	1	1	1	1	1
	2	1	0	1	0	0
	3	1	0	1	0	1
	4	1	1	0	1	1
2. Unit Production	1	10	5	10	15	10
	2	30	0	70	0	0
	3	45	0	75	0	30
	4	40	60	0	60	40
3. Capacity	1	11	6	11	16	11
	2	31	0	61	0	0
	3	46	0	76	0	31
	4	41	61	0	61	41
Total		129	67	158	77	83
Infeasibilities		0	0	18	0	0

Lower Bound Value $2289.00

Cost Per Problem A $2360.00

APPENDIX B

Upper Bounding the CLSP

The heuristic solution procedures for the fixed charge problem described in Appendix C might be used to generate upper bounds for the CLSP. Change constraints (1.2) to (1.2'):

$$\text{(1.2')} \qquad \sum_i r_{ik}^{P} P_{it} \leq R_{KT} - \sum_i r_{ik}^{\delta} \qquad \begin{matrix} k=1,\ldots,K \\ t=1,\ldots,T \end{matrix}$$

A convex polyhedron is now formed by (1.1), (1.2') and P_{it}, $I_{it} > 0$. This new problem (Problem One-A) is more tightly constrained than Problem One so that any feasible solution to (1.0), (1.1), (1.2'), (1.3), (1.4) will be feasible in Problem One. If $\sum_i r_{ik}^{\delta}$ is small compared to R_{KT} and if the heuristic techniques of restricted basis entry simplex method and directed search are applied, a strong upper bound may result. Note that in the sample problem, the best solution of $2045.00 is feasible in Problem One-A, indicating the potential of the bounding procedure for producing tight upper bounds. More specifically, the capacity constraints are reduced by $\sum_i r_{ik}^{\delta} = 4$, from 140 to 136. The capacity requirements (less set-ups) for the $2045.00 solution do not exceed 136. They are 130, 60, 134, 135, and 40.

A good upper bound may not be determined, or even feasible, if $\sum_i r_{ik}^{\delta}$ is large compared to R_{KT}, i.e., when the expression $[R_{KT} - \sum_i r_{ik}^{\delta}]$ is significantly less than R_{KT}.

APPENDIX C

The Fixed Charge Problem (FCP)

If $r^{\delta}_{ik} = 0$ for all i and k, the CLSP is transformed into a general form known as the fixed charge problem.

Consider the problem:

2.0 minimize $\underline{Cx} + \underline{D\delta}(x)$

Subject to:

2.1 $\underline{\underline{A}}\underline{x} = \underline{b}$

2.2 $\underline{x} \geq 0$

2.3 $\underline{\delta}(\underline{x}) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x = 1 \end{cases}$

The extreme point theorem allows us to consider only the extreme points of $Ax = b$, $x \geq 0$ in searching for the optimal solution.

Theorem 3: Let X be a closed convex set which is bounded from above. If the absolute minimum of the concave function f(x) over X is finite, the absolute minimum will be taken at one or more of the extreme points of X.

Proof: (only for Convex Polyhedra. See Hadley (11) for the General Proof.)

Suppose this were not the case, then assume x* is the optimal solution and x* is not an extreme point of $Ax = b$, $x \geq 0$. Then there exist extreme points $x_1, x_2, \ldots x_n$ such that

$f(x_i) > f(x^*)$ for all i and

$$x^* = \sum_i \alpha_i x_i \qquad \alpha_i \geq 0, \quad \sum_i \alpha_i = 1$$

Therefore, by the concavity of f,

$$f(x^*) = f(\sum_i \alpha_i x_i) \geq \sum_i \alpha_i f(x_i) > f(x^*)$$

A contradiction. Therefore the theorem holds.

Though the global minimum occurs at an extreme point, relative minima exist, also at extreme points. Thus it is not possible to use a computational technique of the simplex type (based upon moving from one extreme point to an adjacent one), which terminates when a relative minimum is found. If there is a fixed cost for each variable, every extreme point is a relative minimum (11). Since Problem One has a fixed charge for every second variable, frequent local optima may be expected.

Optimal Solution Procedures for the FCP

There are four procedures suggested - all theoretically optimal but, to varying degrees, computationally disappointing.

1. Total Enumeration of the extreme points. Obviously, for problems of any size, total enumeration is out of the question. The number of extreme points increases combinatorially with m and n.

2. Mixed integer-continuous variable formulation. Hadley (11) presents Gomory's mixed integer formulation of the FCP. Unfortunately, computational experience has been disappointing, even for fairly small problems.

3. Implicit enumeration (Branch and Bound). Steinberg (25) has successfully applied implicit enumeration to the FCP. However, his method is also limited to problems of modest size. If extrapolations are valid, the small sample problem in Appendix A would require the evaluation of 2^{20} nodes.

4. Search by Ranking Extreme Points. Murty (20) solves the FCP by ranking the basic feasible solutions of $Ax = b$, $x \geq 0$ in ascending order of the value of Cx. The applicable fixed charges for the solution are then added less a known minimum fixed charge for any feasible solution, thus specifying an upper bound. This procedure is the most promising of the four, but unfortunately determining the "known minimum value" of the fixed charges for the CLSP is as involved as solving the original problem.

Heuristic Solutions to the Fixed Charge Problem

Heuristic techniques for the FCP exploit the extreme point theorem. We have discussed the limitations of Simplex-like solution procedures, and in an effort to overcome the effect of local minima, the heuristic techniques combine restricted basis entry Simplex methods with directed search.

The heuristics of Denzler (6), Steinberg (25), Cooper and Drebes (3), and Cooper and Olsen (4) are markedly similar. Relative minima are found which are then "perturbed" seeking other (better) relative optima until the procedure is terminated at the best current solution.

The "perturbations" are of two kinds - those which search sequentially outward to adjacent extreme points, and those which "jump" away from the current extreme point. The purpose is to "trickle down" to better local minima. A combination of the two procedures is usually employed, a "jump" occurring when local search has failed to improve the current solution.

Termination occurs when an improved solution is not found after a specified number of consecutive perturbations. No bounding procedures are employed to indicate the degree of non-optimality (if any) of the final solution.

Inapplicability of the FCP to the CLSP

We can relate the FCP to the CLSP on the assumption that the capacity absorption of the setups was zero (equivalent to overnight setups). Unfortunately, if the setups absorb capacity, the constraints no longer form a convex set and the solution procedures for the FCP no longer apply. However, the above heuristics are those suggested in Appendix B as possible tools to determine good upper bounds for the general CLSP.